ETH Zurich

Reading Course

Contiguity

Author:
Peter Hinz

Professor:
Prof. Sara van de Geer

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1 Introduction and Notation

Image Probability Measure Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a random vector $X : \Omega \to \mathbb{R}^n$, then define the *image probability measure* or *pushfoward probability measure* as

$$(X)_*\mathbb{P} : \begin{cases} \mathcal{B}(\mathbb{R})^n & \to [0,1] \\ B & \mapsto \mathbb{P}(X^{-1}(B)) \end{cases}$$

Absolute Continuity Let $\mathbb{P}$ and $\mathbb{Q}$ be two given probability measures on a measurable space $(\Omega, \mathcal{A})$. The measure $\mathbb{Q}$ is said to be *absolutely continuous* with respect to the measure P if $P[A] = 0$ implies $\mathbb{Q}[A] = 0$ for all $A \in \mathcal{A}$. This will be denoted by $\mathbb{Q} \ll \mathbb{P}$. The measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be *mutually orthogonal* if there exists a partition $\Omega = \Omega_\mathbb{P} \bigcup \Omega_\mathbb{Q}$ such that $\Omega_\mathbb{P}$ and $\Omega_\mathbb{Q}$ are disjoint and measurable and $\mathbb{P}(\Omega_\mathbb{Q}) = 0 = \mathbb{Q}(\Omega_\mathbb{P})$. This will be denoted by $\mathbb{P} \perp \mathbb{Q}$.

By the Radon-Nikodym theorem, we can extract a density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ if $\mathbb{Q} \ll \mathbb{P}$. In particular we can define

$$p := \frac{d\mathbb{P}}{d(\frac{1}{2}\mathbb{P} + \frac{1}{1}\mathbb{Q})} \quad q := \frac{d\mathbb{Q}}{d(\frac{1}{2}\mathbb{P} + \frac{1}{1}\mathbb{Q})}$$

Then we can define the measures

$$\mathbb{Q}^a : \begin{cases} \mathcal{A} & \to [0,1] \\ A & \mapsto \mathbb{Q}(A \cap \{p > 0\}) \end{cases} \quad \mathbb{Q}^\perp : \begin{cases} \mathcal{A} & \to [0,1] \\ A & \mapsto \mathbb{Q}(A \cap \{p = 0\}) \end{cases}$$

It is easy to see that the following holds:

- $\mathbb{Q}^a \ll \mathbb{P}$ (If $\mathbb{P}(A) = 0$ then $\int_\Omega \mathbb{1}_A p \, d(\frac{1}{2}\mathbb{P} + \frac{1}{2}\mathbb{Q})$ then $\mathbb{1}_A \cdot p = 0$ $\mathbb{Q}$-a.s.)
- $Q^\perp \perp \mathbb{P}$ (take $\Omega_{Q^\perp} = \{p = 0\}$ and $\Omega_\mathbb{P} = \{p = 0\}$)
- $\mathbb{Q} = \mathbb{Q}^a + \mathbb{Q}^\perp$
- since $\int_\Omega \mathbb{1} A \cdot \frac{q}{p} d\mathbb{P} = \int_\Omega \mathbb{1}_A \cdot q \cdot \mathbb{1}_{\{p>0\}} \, d(\frac{1}{2}\mathbb{P} + \frac{1}{2}\mathbb{Q})$

$$\forall A \in \mathcal{A} : \; \mathbb{Q}^a(A) = \int_\Omega \mathbb{1}_A \cdot \frac{q}{p} d\mathbb{P} \tag{1}$$

- For the next section, it is of special interest that

$$\mathbb{Q} \ll \mathbb{P} \iff \mathbb{Q}^\perp = 0 \iff \mathbb{Q}^a = \mathbb{Q},$$

  in other words

$$\mathbb{Q} \ll \mathbb{P} \iff \mathbb{Q}(\{p = 0\}) = 0 \iff \int_\Omega \frac{q}{p} d\mathbb{P} = 1. \tag{2}$$

Important Remark On Notation Even if $\mathbb{Q} \ll \mathbb{P}$ doesn't hold, we will use the notation

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{d\mathbb{Q}^a}{d\mathbb{P}}.$$

From (1), it follows that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{q}{p} \quad \mathbb{P}\text{-almost surely.}$$

Furthermore, it then holds for nonnegative measurable f that

$$\int_\Omega f d\mathbb{Q} \geq \int_\Omega f d\mathbb{Q}^a = \int_\Omega f \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} \tag{3}$$

and in this notation, (2) can be written as

$$\mathbb{Q} \ll \mathbb{P} \iff \mathbb{Q}(\{\tfrac{d\mathbb{P}}{d\mathbb{Q}} = 0\}) = 0 \iff \int_\Omega \tfrac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} = 1. \tag{4}$$

Change of measure If $\mathbb{Q} \ll \mathbb{P}$, then for each random vector $X : \Omega \to \mathbb{R}^k$ the $\mathbb{Q}$-law of X can be obtained by the $\mathbb{P}$-law of $(X, \frac{d\mathbb{Q}}{d\mathbb{P}})$:

$$\begin{aligned} \forall A \in \mathcal{B}(\mathbb{R})^k: \quad (X_*\mathbb{Q})(A) &= \int_\Omega \mathbb{1}_A(X(\omega))\, d\mathbb{Q}(\omega) = \int_\Omega \mathbb{1}_A(X(\omega)) \tfrac{d\mathbb{Q}}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega) \\ &= \int_{\mathbb{R}^k \times \mathbb{R}} \mathbb{1}_A(x) v \, d(X, \tfrac{d\mathbb{Q}}{d\mathbb{P}})_*\mathbb{P}(x, v) \end{aligned} \tag{5}$$

2 Contiguity

Definition 1 (Contiguity)**.** Let $(\Omega_n, \mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of measurable spaces. For each $n \in \mathbb{N}$ assume there are two probability measures $\mathbb{Q}_n$ and $\mathbb{P}_n$ on $(\Omega_n, \mathcal{A}_n)$. The sequence $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ is *contiguous* with respect to the sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ if

$$\forall \, (A_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{A}_n : \quad \lim_{n\to\infty} \mathbb{P}_n(A_n) = 0 \implies \lim_{n\to\infty} \mathbb{Q}_n(A_n) = 0.$$

This will be denoted by $(\mathbb{Q}_n)_{n\in\mathbb{N}} \triangleleft (\mathbb{P}_n)_{n\in\mathbb{N}}$. If $(\mathbb{Q}_n)_{n\in\mathbb{N}} \triangleleft (\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(\mathbb{P}_n)_{n\in\mathbb{N}} \triangleleft (\mathbb{Q}_n)_{n\in\mathbb{N}}$ then $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ are *mutually contiguous*, which will be denoted by

$$(\mathbb{P}_n)_{n\in\mathbb{N}} \triangleleft \triangleright (\mathbb{Q}_n)_{n\in\mathbb{N}}.$$

Theorem 2. *The sets of measures* $((\frac{d\mathbb{Q}_n}{d\mathbb{P}_n})_* d\mathbb{P}_n)_{n\in\mathbb{N}}$ *and* $((\frac{d\mathbb{P}_n}{d\mathbb{Q}_n})_*\mathbb{Q}_n)_{n\in\mathbb{N}}$ *are tight.*

Proof. The densities are nonnegative and their expectations are bounded by 1. Tightness follows from

$$(\tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n})_*\mathbb{P}_n \left[[M, \infty)\right] \leq \frac{1}{M} \int_\mathbb{R} \tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n} d\mathbb{P}_n \leq \frac{1}{M}, \qquad (\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n})_*\mathbb{Q}_n \left[[M, \infty)\right] \leq \frac{1}{M} \int_\mathbb{R} \tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n} d\mathbb{Q}_n \leq \frac{1}{M},$$

using the fact that for a probability measure μ on $(\Omega, \mathcal{A})$ and a nonnegative random variable $X : \Omega \to \mathbb{R}$ it holds

$$\forall M \geq 0 : \quad M\mu(\{X \geq M\}) \leq \int_\Omega X d\mu.$$

□

The following theorem is often called Le Cam's first lemma. It gives two additional characterizations when there is contiguity. It can be seen as an asymptotic version of (4).

Theorem 3 (Le Cam's first lemma)**.** *Let* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *and* $(\mathbb{Q}_n)_\mathbb{N}$ *be sequences of probability measures on measurable spaces* $(\Omega_n, \mathcal{A}_n)_{n\in\mathbb{N}}$*. The following are equivalent.*

1. $(\mathbb{Q}_n)_{n\in\mathbb{N}} \lhd (\mathbb{P}_n)_{n\in\mathbb{N}}$

2. If $(d\mathbb{P}_n/d\mathbb{Q}_n)_*\mathbb{Q}_n \to \mathbb{U}$ *in distribution along a subsequence for some probability measure* $\mathbb{U}$ *on* $\mathbb{R}$ *then*

$$\mathbb{U}((0,\infty)) = 1$$

3. If $(d\mathbb{Q}_n/d\mathbb{P}_n)_*\mathbb{P}_n \to \mathbb{V}$ *in distribution along a subsequence for some probability measure* $\mathbb{V}$ *on* $\mathbb{R}$*, then*

$$\int_{\mathbb{R}} x d\mathbb{V}(x) = 1$$

Proof.

1. implies 2. Assume that there is some subsequence along which

$$(\tfrac{d\mathbb{P}_{n_k}}{d\mathbb{Q}_{n_k}})_*\mathbb{Q}_{n_k} \to \mathbb{U} \text{ in distribution as } k\to\infty.$$

Then it follows by the Portmanteau Theorem 9

$$\forall\varepsilon>0: \quad \liminf_{k\to\infty} (\tfrac{d\mathbb{P}_{n_k}}{d\mathbb{Q}_{n_k}})_*\mathbb{Q}_{n_k}\left((-\infty,\varepsilon)\right) - \mathbb{U}\left((-\infty,\varepsilon)\right) \geq 0.$$

For a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to 0 sufficiently slowly, this implies

$$\liminf_{k\to\infty} (\tfrac{d\mathbb{P}_{n_k}}{d\mathbb{Q}_{n_k}})_*\mathbb{Q}_{n_k}\left((-\infty,\varepsilon_{n_k})\right) - \mathbb{U}\left((-\infty,\varepsilon_{n_k})\right) \geq 0$$

or equivalently

$$\mathbb{U}\left((-\infty,0]\right) \leq \liminf_{k\to\infty} (\tfrac{d\mathbb{P}_{n_k}}{d\mathbb{Q}_{n_k}})_*\mathbb{Q}_{n_k}\left((-\infty,\varepsilon_{n_k})\right). \tag{6}$$

It also holds[1]

$$\mathbb{P}_n\left(\left\{(\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n}) < \varepsilon_n \wedge q_n > 0\right\}\right) = \int_{\frac{d\mathbb{P}_n}{d\mathbb{Q}_n}<\varepsilon_n} \frac{d\mathbb{P}_n}{d\mathbb{Q}_n} d\mathbb{Q}_n \leq \int_{\Omega_n} \varepsilon_n d\mathbb{Q}_n \to 0 \text{ as } n\to\infty$$

and now the contiguity assumption 1. implies that

$$\mathbb{Q}_n\left(\left\{(\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n}) < \varepsilon_n\right\}\right) = \mathbb{Q}_n\left(\left\{(\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n}) < \varepsilon_n \wedge q_n > 0\right\}\right) \to 0 \text{ as } n\to\infty.$$

Combined with (6), this implies $\mathbb{U}\left((0,\infty)\right) = 1$.

2. implies 3. Assume now that 2. holds and let some subsequence $(n_k)_{k\in\mathbb{N}}$ be given such that

$$(\tfrac{d\mathbb{Q}_{n_k}}{d\mathbb{P}_{n_k}})_*\mathbb{P}_{n_k} \to \mathbb{V} \text{ in distribution as } k\to\infty$$

for some probability measure $\mathbb{V}$ on $\mathbb{R}$. Define the probability measures

$$\mu_n = \frac{1}{2}(\mathbb{P}_n + \mathbb{Q}_n) \text{ for } n\in\mathbb{N}.$$

Then for all $n\in\mathbb{N}$ we have $\mathbb{P}_n \ll \mu_n$ and $\mathbb{Q}_n \ll \mu_n$ and the densities $W_n = \frac{d\mathbb{P}_n}{d\mu_n}$ and $R_n = \frac{d\mathbb{Q}_n}{d\mu_n}$ satisfy

$$\forall n\in\mathbb{N}: \quad R_n + W_n = 2 \ \mu_n\text{-almost surely.}$$

[1] Let $q_n = \frac{d\mathbb{Q}_n}{d(\frac{1}{2}\mathbb{Q}_n + \frac{1}{2}\mathbb{P}_n)}$, see first Section 1.

Since the densities R_n and W_n are thus in $[0,2]$ μ_n-almost surely for every $n \in \mathbb{N}$, the sequence of Probability measure $(W_n)_*\mu_n$ are tight and Prohorov's Theorem 8 can be used together with Theorem 2 to find a further subsequence $(n_{k_j})_{j\in\mathbb{N}}$ such that

$$(\frac{dQ_{n_{k_j}}}{d\mathbb{P}_{n_{k_j}}})_*\mathbb{P}_{n_{k_j}} \to \mathbb{V}, \quad (\frac{dP_{n_{k_j}}}{d\mathbb{Q}_{n_{k_j}}})_*\mathbb{Q}_{n_{k_j}} \to \mathbb{U}, \quad (W_{n_{k_j}})_*\mu_{n_{k_j}} \to \mathbb{W}$$

as $j \to \infty$ for some probability measures $\mathbb{U}, \mathbb{W} : \mathcal{B}(\mathbb{R}) \to [0,1]$. Since the densities $(W_n)_{n\in\mathbb{N}}$ take values in $[0,2]$ μ_n-almost surely, the probability measures $((W_n)_*\mu_n)_{n\in\mathbb{N}}$ are asymptotically uniformly bounded and Theorem 12 implies the convergence of moments:

$$\int_{\mathbb{R}} x d\mathbb{W}(x) = \lim_{n\to\infty} \int_{\mathbb{R}} \frac{d\mathbb{P}_n}{d\mu_n}(x) d\mu_n(x) = 1 \tag{7}$$

For a given function $f : \mathbb{R} \to \mathbb{R}$ define the functions g_f and h_f by

$$g_f : \begin{cases} \mathbb{R} & \to \mathbb{R} \\ x & \mapsto \begin{cases} 2f(0) & \text{if } x < 0 \\ f(\frac{x}{2-x})(2-x) & \text{if } x \in [0,2) \\ 0 & \text{if } x \geq 2 \end{cases} \end{cases}$$

$$h_f : \begin{cases} \mathbb{R} & \to \mathbb{R} \\ x & \mapsto \begin{cases} 0 & \text{if } x \leq 0 \\ f(\frac{2-x}{x})x & \text{if } x \in (0,2] \\ 2f(0) & \text{if } x > 2 \end{cases} \end{cases}$$

It is useful to collect some facts:

- If $|f|$ is bounded by $c < \infty$ then $|g_f|$ and $|h_f|$ are bounded by $2c$.
- If f is bounded and continuous then g_f and h_f are bounded and continuous
- If $f \geq 0$ then $g_f \geq 0$ and $h_f \geq 0$.
- If a sequence $(f_n)_{n\in\mathbb{N}}$ of functions converges pointwise to a function f then g_{f_n} and h_{f_n} converge pointwise to g_f and h_f respectively.

These facts will be used to apply the monotone and dominated convergence theorems. Now let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, so the same holds for g_f and h_f. From the definition of μ_n, it follows that

$$\frac{d\mathbb{P}_n}{d\mathbb{Q}_n} = \frac{W_n}{2-W_n} \quad \mathbb{Q}_n\text{-a.s.}, \qquad \frac{d\mathbb{Q}_n}{d\mathbb{P}_n} = \frac{2-W_n}{W_n} \quad \mathbb{P}_n\text{-a.s.}$$

and μ_n-almost surely

$$\frac{d\mathbb{Q}_n}{d\mu_n} = 2 - W_n, \quad \frac{d\mathbb{P}_n}{d\mu_n} = W_n.$$

Therefore, it holds

$$\begin{aligned}
\int_{\mathbb{R}} f(x) d(\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n})_* \mathbb{Q}_n(x) &= \int_{\mathbb{R}} f(\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n}(x)) d\mathbb{Q}_n(x) \\
&= \int_{\mathbb{R}} f(\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n}(x)) \tfrac{d\mathbb{Q}_n}{d\mu_n}(x) d\mu_n(x) \\
&= \int_{\mathbb{R}} g_f(W_n) d\mu_n(x) \\
\int_{\mathbb{R}} f(x) d(\tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n})_* \mathbb{P}_n(x) &= \int_{\mathbb{R}} f(\tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n}(x)) d\mathbb{P}_n(x) \\
&= \int_{\mathbb{R}} f(\tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n}(x)) \tfrac{d\mathbb{P}_n}{d\mu_n}(x) d\mu_n(x) \\
&= \int_{\mathbb{R}} h_f(W_n(x)) d\mu_n(x).
\end{aligned}$$

Taking limits yields by the Portmanteau Theorem 9

$$\int_{\mathbb{R}} f(x) d\mathbb{U}(x) = \int_{\mathbb{R}} g_f(x) d\mathbb{W}(x), \quad \int_{\mathbb{R}} f(x) d\mathbb{V}(x) = \int_{\mathbb{R}} h_f(x) d\mathbb{W}(x).$$

For a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions converging pointwise to $\mathbb{1}_{\{0\}}$ with $|f_n| \leq 1$ for $n \in \mathbb{N}$, the facts above imply that $(g_{f_n})_{n\in\mathbb{N}}$ is a sequence of continuous functions converging pointwise to $2\mathbb{1}_{(-\infty,0]}$ with $|g_{f_n}| \leq 2$ for $n \in \mathbb{N}$ and the dominated convergence theorem can be applied:

$$\mathbb{U}\left[\{0\}\right] = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(d) dU(x) = \lim_{n\to\infty} \int_{\mathbb{R}} g_{f_n}(x) d\mathbb{W}(x) = 2\mathbb{W}\left[(-\infty, 0]\right] = 2\mathbb{W}[\{0\}] \quad (8)$$

Similarly for an increasing sequence of continuous functions converging pointwise to $x \mapsto \mathbb{1}_{[0,\infty)}(x)x$ with $f_n \geq 0$ for $n \in \mathbb{N}$, the facts above imply that $(h_{f_n})_{n\in\mathbb{N}}$ is an increasing sequence of continuous functions converging pointwise to $x \mapsto \mathbb{1}_{(0,2]}(x) \cdot (2-x)$ with $h_{f_n} \geq 0$ for $n \in \mathbb{N}$ and the monotone convergence Theorem can be applied:

$$\int_{\mathbb{R}} x d\mathbb{V}(x) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) d\mathbb{V}(x) = \lim_{n\to\infty} \int_{\mathbb{R}} h_{f_n}(x) d\mathbb{W}(x) = 2 - 2\mathbb{W}\left[\{0\}\right] - \int_{\mathbb{R}} x d\mathbb{W}(x)$$

Combining this with equations (7) and (8) completes the proof:

$$\mathbb{U}\left[\{0\}\right] + \int_{\mathbb{R}} x d\mathbb{V}(x) = 1$$

3. implies 1. Let $(A_n)_{n\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathcal{A}_n$ such that

$$\lim_{n\to\infty} \mathbb{P}_n\left(A_n\right) = 0$$

and assume by contradiction that

$$\limsup_{n\to\infty} \mathbb{Q}_n(A_n) > \varepsilon$$

for some $\varepsilon > 0$. By taking a subsequence, without loss of generality assume that

$$\forall n \in \mathbb{N} : \quad \mathbb{Q}_n(A_n) > \varepsilon.$$

Obviously $(\mathbb{1}_{\Omega_n\setminus A_n})_*\mathbb{P}_n \to \delta_1$ in distribution[2]. Take a limit in distribution $\mathbb{V}$ of a subsequence of $(\frac{d\mathbb{Q}_n}{d\mathbb{P}_n})_*\mathbb{P}_n$ (there exist such limits by Prohorov's Theorem 8 and Theorem 2).

$$(\tfrac{d\mathbb{Q}_{n_k}}{d\mathbb{P}_{n_k}})_*\mathbb{P}_{n_k} \to \mathbb{V} \text{ in distribution as } k\to\infty.$$

It is easy to see that this implies

$$(\tfrac{d\mathbb{Q}_{n_k}}{d\mathbb{P}_{n_k}}, \mathbb{1}_{\Omega_{n_k}\setminus A_{n_k}})_*\mathbb{P}_{n_k} \to \mathbb{V}\otimes\delta_1 \text{ in distribution as } k\to\infty,$$

where $\mathbb{V}\otimes\delta_1$ denotes the product measure of $\mathbb{V}$ and δ_1. Since $\mathbb{R}^2\to\mathbb{R}, (s,t)\mapsto s\cdot t\cdot \mathbb{1}_{[0,\infty)\times[0,\infty)}(s,t)$ is a continuous nonnegative function, the Portmanteau Theorem 9 implies the second to last step in the following calculation:

$$\begin{aligned}
1-\varepsilon &\geq 1-\limsup_{k\to\infty}\mathbb{Q}_{n_k}(A_{n_k})\\
&\geq \liminf_{k\to\infty}\mathbb{Q}_{n_k}(\Omega_{n_k}\setminus A_{n_k})\\
&\geq \liminf_{k\to\infty}\int_{\Omega_{n_k}} \mathbb{1}_{\Omega_{n_k}\setminus A_{n_k}}\frac{d\mathbb{Q}_{n_k}}{d\mathbb{P}_{n_k}}d\mathbb{P}_{n_k}\\
&= \liminf_{k\to\infty}\int_{\mathbb{R}^2} s\cdot t\cdot\mathbb{1}_{[0,\infty)\times[0,\infty)}(s,t)\; d(\tfrac{\mathbb{Q}_{n_k}}{d\mathbb{P}_{n_k}}, \mathbb{1}_{\Omega_{n_k}\setminus A_{n_k}})_*\mathbb{P}_{n_k}(s,t)\\
&\geq \int_{\mathbb{R}^2} s\cdot t\; d\mathbb{V}\otimes\delta_1(s,t) = \int_{\mathbb{R}} s d\mathbb{V}(s) = 1,
\end{aligned}$$

which is a contradiction to $\varepsilon > 0$. □

Example 4. *Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ be probability measures on measurable spaces $(\Omega_n,\mathcal{F}_n)_{n\in\mathbb{N}}$ such that*[3]

$$(\tfrac{d\mathbb{P}_n}{d\mathbb{Q}_n})_*\mathbb{Q}_n \to e^{\mathcal{N}(\mu,\sigma^2)} \textit{ in distribution as } n\to\infty$$

Then Le Cam's first lemma allows us to conclude

- $(\mathbb{Q}_n)_{n\in\mathbb{N}} \triangleleft (\mathbb{P}_n)_{n\in\mathbb{N}}$ *because* $e^{\mathcal{N}(\mu,\sigma^2)}[(0,\infty)] = 1$.
- *If* $\mu = -\frac{1}{s}\sigma^2$ *then* $(\mathbb{P}_n)_{n\in\mathbb{N}}\triangleleft(\mathbb{Q}_n)_{n\in\mathbb{N}}$ *because* $\int_{\mathbb{R}} x de^{\mathcal{N}(\mu,\sigma^2)}(x) = e^{\mu+\frac{1}{2}\sigma^2} = 1$. *Thus, in this case $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ are mutually contiguous.*

The following theorem shows how a $\mathbb{Q}_n$ limit law can be obtained from a $\mathbb{P}_n$ limit law. This is analogous to the non asymptotic case in equation (5).

Theorem 5 (Le Cam's third lemma)**.** *Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ be probability measures on measurable spaces $(\Omega_n,\mathcal{A}_n)_{n\in\mathbb{N}}$ with random vectors $(X_n:\Omega_n\to\mathbb{R}^k)_{n\in\mathbb{N}}$. Assume that $(\mathbb{Q}_n)_{n\in\mathbb{N}}\triangleleft(\mathbb{P}_n)_{n\in\mathbb{N}}$ and*

$$\left(X_n, \tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n}\right)_*\mathbb{P}_n \to (X,V)_*\mu \textit{ in distribution as } n\to\infty$$

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[2] Let $\delta_1 : \begin{cases}\mathcal{B}(\mathbb{R}) & \to [0,1]\\ B & \mapsto \mathbb{1}_B(1)\end{cases}$

[3] Let $e^{\mathcal{N}(\mu,\sigma^2)}$ denote the image probability measure of a $\mathcal{N}(\mu,\sigma^2)$ distribution under the map $x\mapsto e^x$ (log-normal distribution)

for some probability measure μ on some measurable space $(\Omega, \mathcal{A})$ with random variables $X : \Omega \to \mathbb{R}^k$ and $V : \Omega \to \mathbb{R}$. Then

$$\mathbb{L} : \begin{cases} \mathcal{B}(\mathbb{R})^k & \to [0,1] \\ B & \mapsto \int_\Omega \mathbb{1}_B(X(\omega))V(\omega)d\mu(\omega) \end{cases}$$

is a probability measure and satisfies that

$$(X_n)_*\mathbb{Q}_n \to \mathbb{L} \text{ in distribution as } n \to \infty.$$

Proof. $\mathbb{L}$ is a measure since sigma-additivity follows from the monotone convergence theorem. It is a probability measure since $\mathbb{L}(\mathbb{R}^k) = \int_\mathbb{R} x dV_*\mu(x) = 1$ because of Theorem 3 (Note that $(\frac{d\mathbb{Q}_n}{d\mathbb{P}_n})_*\mathbb{P}_n \to (V)_*\mu$ in distribution). For $f = \mathbb{1}_B$ with $B \in \mathcal{B}(\mathbb{R})^k$, it is clear that

$$\int_\mathbb{R} f d\mathbb{L} = \int_\Omega f(X(\omega))V(\omega)d\mu(\omega),$$

so this holds for step functions f, hence for positive measurable functions by monotone convergence, hence for integrable functions f by dominated convergence.
For a continuous nonnegative function $f : \mathbb{R}^k \to \mathbb{R}$, the function $(x, v) \mapsto f(x)v\mathbb{1}_{[0,\infty)}(v)$ is continuous and nonnegative. The Portmanteau Theorem 9 implies the third step in the following calculation.

$$\begin{aligned} \liminf_{n\to\infty} \int_{\mathbb{R}^k} f(x)d(X_n)_*\mathbb{Q}_n(x) &\geq \liminf_{n\to\infty} \int_{\Omega_n} f(X_n(\omega))\tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n}(\omega)d\mathbb{P}_n(\omega) \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^{k+1}} f(x)v \; d(X_n, \tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n})_*\mathbb{P}_n(x, v) \\ &\geq \int_{\mathbb{R}^k} f(x)v \; d(X, V)_*\mu(x, v) \\ &= \int_{\mathbb{R}^k} f d\mathbb{L} \end{aligned}$$

This inequality holds for every continuous and nonnegative function $f : \mathbb{R}^k \to \mathbb{R}$ and hence the Portmanteau Theorem 9 implies that

$$(X_n)_*\mathbb{Q}_n \to \mathbb{L} \text{ in distribution as } n \to \infty.$$

□

The following result is a consequence of Le Cam's third lemma but often the term "Le Cam's third lemma" refers to this special case.

Example 6. *Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ be probability measures on measurable spaces $(\Omega_n, \mathcal{A}_n)_{n\in\mathbb{N}}$ with random vectors $(X_n : \Omega_n \to \mathbb{R}^k)_{n\in\mathbb{N}}$. Assume that*

$$(X_n, \log(\tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n}))_*\mathbb{P}_n \to \mathcal{N}\left(\begin{pmatrix} m \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^T & \sigma^2 \end{pmatrix}\right) =: \mu \text{ in distribution as } n \to \infty$$

For notational convenience define the maps

$$X : \begin{cases} \mathbb{R}^{k+1} & \to \mathbb{R}^k \\ (x_1, \ldots, x_{k+1}) & \mapsto (x_1, \ldots, x_k) \end{cases} \text{ and } W : \begin{cases} \mathbb{R}^{k+1} & \to \mathbb{R} \\ (x_1, \ldots, x_{k+1}) & \mapsto x_{k+1} \end{cases}.$$

By the Continuous Mapping Theorem 10

$$(X_n, \tfrac{d\mathbb{Q}_n}{d\mathbb{P}_n}) \to (X, e^W)_*\mu \text{ in distribution as } n \to \infty.$$

By the last example, $(\mathbb{P}_n)_{n\in\mathbb{N}} \lhd\rhd (\mathbb{Q}_n)_{n\in\mathbb{N}}$ and we can apply the theorem above. It states that

$$(X_n)_*\mathbb{Q}_n \to \mathbb{L} \text{ in distribution as } n \to \infty,$$

where $\mathbb{L}$ is defined as

$$\mathbb{L} : \begin{cases} \mathcal{B}(\mathbb{R})^k & \to [0,1] \\ B & \mapsto \int_{\mathbb{R}^k\times\mathbb{R}} \mathbb{1}_B(X(x)) \cdot e^{W(x)} d\mu(x) \end{cases}$$

For the characteristic functions $\Phi_{\mathbb{L}} : \mathbb{R}^k \to \mathbb{C}$ and $\Phi_\mu : \mathbb{R}^{k+1} \to \mathbb{C}$ holds

$$\begin{aligned} \Phi_{\mathbb{L}}(t) &= \int_{\mathbb{R}^k} e^{it^T x} d\mathbb{L}(x) = \int_{\mathbb{R}^{k+1}} e^{it^T X(x)} e^{W(x)} d\mu(x) = \Phi_\mu((t, -i)) \\ &= \exp\left(it^T m - \frac{1}{2}\sigma^2 - \frac{1}{2}(t^T, -i) \begin{pmatrix} \Sigma & \tau \\ \tau^T & \sigma^2 \end{pmatrix} \begin{pmatrix} t \\ -i \end{pmatrix} \right) \\ &= \exp\left(it^T(m+\tau) - \frac{1}{2} t^T \Sigma t \right) \end{aligned}$$

This means that

$$(X_n)_*\mathbb{Q}_n \to \mathbb{L} = \mathcal{N}\left((m+\tau), \Sigma\right) \text{ in distribution as } n \to \infty.$$

A Appendix

For the following theorems and definitions refer to [1]. In the sequel, let $d \in \mathbb{N}$.

Definition 7 (Tightness)**.** A sequence of probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ on $\mathbb{R}^d$ is tight if

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}_n \left(\left\{ x \in \mathbb{R}^d | \, \|x\| \geq M \right\} \right) = 0$$

Theorem 8 (Prohorov)**.** *Let (μ_n) be a sequence of probability measures on $\mathbb{R}^d$.*

1. *If $\mu_n \to \mu$ in distribution for some probability measure μ, then $\{\mu_n | n \in \mathbb{N}\}$ is tight.*
2. *If $(\mu_n)_{n\in\mathbb{N}}$ is tight, then there exists a subsequence $(\mu_{n_j})_{j\in\mathbb{N}}$ with $\mu_{n_j} \to \mu$ as $j \to \infty$ for some probability measure μ on $\mathbb{R}^d$.*

The following Theorem states parts of the Portmanteau Theorem.

Theorem 9 (Portmanteau)**.** *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on $\mathbb{R}^d$. The following are equivalent*

1. *$\mu_n \to \mu$ in distribution*
2. *$\lim_{n\to\infty} \int_{\mathbb{R}^d} f d\mu_n = \int_{\mathbb{R}^d} f d\mu$ for all bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$.*
3. *$\liminf_{n\to\infty} \int_{\mathbb{R}^d} f d\mu_n \geq \int_{\mathbb{R}^d} f d\mu$ for all nonnegative continuous functions $f : \mathbb{R}^d \to \mathbb{R}$.*

4. $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ *for every open set* $U \subset \mathbb{R}^d$.

Theorem 10 (Continuous Mapping)**.** *Let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence of probability measures on* $\mathbb{R}^d$ *converging in distribution to a probability measure* μ *and let* $g : \mathbb{R}^d \to \mathbb{R}^m$ *be continuous at every point of a set* C *such that* $\mu[C] = 1$ *Then*

$$g_*\mu_n \to g_*\mu \text{ in distribution as } n \to \infty.$$

Definition 11 (Asymptotic Uniform Integrability)**.** A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}^d$ is asymptotically uniformly integrable if

$$\lim_{M\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^d} \|x\| \mathbb{1}_{\{x\in\mathbb{R}^d |\ \|x\|_2 > M\}} d\mu_n(x) = 0$$

Theorem 12 (Bounded Convergence)**.** *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be measurable and continuous at every point in a set* C*. Let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence of probability measures on* $\mathbb{R}^d$ *converging in distribution to some measure* μ *on* $\mathbb{R}^d$ *with total mass contained in* C*. If the sequence* $((f)_*\mu_n)_{n\in\mathbb{N}}$ *is asymptotically uniformly integrable, then*

$$\int_{\mathbb{R}^d} f(x) d\mu_n(x) \to \int_{\mathbb{R}^d} f(x) d\mu(x)$$

References

[1] A.W. van der Vaart. Asymptotic Statistics. Cambridge University Press, 1998.